Scattering of Surface Waves by Discontinuities on a Unidirectionally Conducting Screen*

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Summary—It is shown that a plane screen consisting of closely-spaced parallel wires which are separated from one another and which are such that the radius of the wires and the spacing between them are small in comparison to wavelength, can support a surface wave, the spread of whose field components depends only on the angle which the direction of propagation makes with the direction of the wires. The problem of radiation from a discontinuity in such a semi-infinite waveguide is studied for the following three types of discontinuities: 1) when the waveguide terminates in empty space, 2) when it terminates at another such semi-infinite waveguide having different propagation characteristics, and 3) when it terminates at a perfectly conducting half-plane. In each case, the power reflection coefficient, where applicable the power transmission coefficient, the loss of power due to radiation, and its angular distribution are evaluated. The motivation for this investigation is briefly indicated.

Introduction

THE PROPAGATION of electromagnetic waves in waveguides with anisotropic walls has in recent years assumed practical importance in long-distance waveguide communication.[1,2] One such waveguide which is commonly used is in the form of a tightly-wound helix, in which the adjacent turns are separated from each other. In this connection it is of interest to investigate theoretically the effect on the propagation of guided waves introduced by the junction formed by either two different anisotropic (helical) waveguides or a helical waveguide and a circular waveguide having perfectly conducting walls. Also, helical waveguides of finite length, known commonly as helical antennas, are widely used to obtain radiation along the axis of the helix. In view of this practical application the investigation of radiation from the open end of a helical waveguide is also of interest.

As a first step in the understanding of the more difficult problem of radiation from discontinuities in a helical waveguide, it is advantageous to treat the limiting case in which the radius of the helix becomes infinite. For the limiting case, the helical waveguide degenerates into a plane screen that is conducting only in the direction of the wires composing it, and insulating in the perpendicular direction. In this paper, a treatment is given for the problem of radiation from discontinuities in such a planar waveguide consisting of parallel wires which are separated from each other and which are such that the radius of the wires and the spacing between them are quite small compared to wavelength.

In the first section, it is shown that an anisotropic planar surface can support a guided wave which is attenuated exponentially in the direction normal to the surface. The spread of the field in this surface waveguide decreases as the angle between the direction of propagation and the direction of the wires becomes close to $\pi/2$. Appropriate boundary conditions applicable at the surface of such a unidirectionally conducting screen have been given recently by Karp.[3] Radiation from the open end of such a semi-infinite surface waveguide is treated in the next section. Expressions for the power reflection coefficient and the radiation pattern are obtained.

In the third section, the electromagnetic fields produced at the junction of two semi-infinite surface waveguides are examined; the wires composing the two surface waveguides are assumed to be in different directions. It is important to note that by a suitable formulation this problem is reduced to the question of a solution of a functional equation similar to the one studied previously by Kay.[4]

A treatment is given in the final section for the problem of radiation from the junction formed by a semi-infinite surface waveguide and a perfectly conducting half-plane. For this case, the power reflection coefficient and the radiation pattern are found to be the same as for the open-ended waveguide.

In an earlier paper,[5] the problem of excitation of surface waves on a unidirectionally conducting screen has been investigated.

Surface Wave on a Unidirectionally Conducting Screen

Consider a unidirectionally conducting screen occupying the region $-\infty \leq x \leq \infty$, $-\infty \leq y \leq \infty$, and $z=0$, where x, y, z form a right-handed rectangular coordinate

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[1] A. E. Karbowiak, "Microwave propagation in anisotropic waveguides," *Proc. IEE*, vol. 103C, pp. 139–144; August, 1955. (Monograph No. 147R.)

[2] A. E. Karbowiak, "Microwave aspects of waveguides for long distance transmission," *Proc. IEE*, vol. 105C, pp. 360–369; February, 1958. (Monograph No. 287R.)

[3] S. N. Karp, "Diffraction of a Plane Wave by a Unidirectionally Conducting Half-Plane," Inst. Math. Sci., New York University, N. Y., Res. Rep. No. EM-108; August, 1957.

[4] A. F. Kay, "Scattering of a surface wave by a discontinuity in reactance," IRE Trans. on Antennas and Propagation, vol. AP-7, pp. 22–31; January, 1959.

[5] S. R. Seshadri, "Excitation of surface waves on a unidirectionally conducting screen," IRE Trans. on Microwave Theory and Techniques, vol. 10, pp. 279–286; July, 1962.

system. Also set up two rotated coordinate systems (ξ_1, η_1, z) and (ξ_2, η_2, z) where

$$\begin{aligned} \xi_{1,2} &= x \cos \alpha_{1,2} + y \sin \alpha_{1,2} \\ \eta_{1,2} &= -x \sin \alpha_{1,2} + y \cos \alpha_{1,2} \\ z &= z \qquad 0 \leq \alpha_{1,2} < \pi/2. \end{aligned} \tag{1}$$

The screen is assumed to be conducting in the direction ξ_1 only. The electromagnetic fields $\boldsymbol{E}$, $\boldsymbol{H}$ satisfy the time harmonic Maxwell's equations

$$\begin{aligned} \nabla \times \boldsymbol{E} &= ik\boldsymbol{H} \\ \nabla \times \boldsymbol{H} &= -ik\boldsymbol{E} \end{aligned} \tag{2}$$

in the region exterior to the screen. The harmonic time dependence $e^{-i\omega t}$ is implied for all the field components. On the screen the following boundary conditions are satisfied

$$E_{\xi_1}(x, y, o) = 0 \tag{3}$$

$$H_{\xi_1}(x, y, o^+) - H_{\xi_1}(x, y, o^-) = 0 \tag{4}$$

$$E_{\eta_1}(x, y, o^+) - E_{\eta_1}(x, y, o^-) = 0. \tag{5}$$

Let the mode for which $H_{\xi_1}=0$ be considered. All the field components are conveniently derived using the electric vector potential $\boldsymbol{A}$, which, since $H_{\xi_1}=0$, is entirely in the ξ_1 direction. The geometry of the screen itself is independent of the y-coordinate, hence it is reasonable to look for field components which do not vary with y. Because of(1), it is seen that

$$\frac{\partial}{\partial \xi_1} = \cos \alpha_1 \frac{\partial}{\partial x} \qquad \frac{\partial}{\partial \eta_1} = -\sin \alpha_1 \frac{\partial}{\partial x}\cdot \tag{6}$$

Also since

$$\boldsymbol{H} = \nabla \times \hat{\xi}_1 A \tag{7}$$

$$\boldsymbol{E} = -\frac{1}{ik} \nabla \times \nabla \times \hat{\xi}_1 A, \tag{8}$$

it follows that

$$\begin{aligned} H_{\xi_1} &= 0 \\ H_{\eta_1} &= \frac{\partial}{\partial z} A(x, z) \\ H_z &= \sin \alpha_1 \frac{\partial}{\partial x} A(x, z) \\ E_{\xi_1} &= \frac{i}{k}\left(k^2 + \cos^2 \alpha_1 \frac{\partial^2}{\partial x^2}\right) A(x, z) \\ E_{\eta_1} &= -\frac{i}{k} \cos \alpha_1 \sin \alpha_1 \frac{\partial^2}{\partial x^2} A(x, z) \\ E_z &= \frac{i}{k} \cos \alpha_1 \frac{\partial^2}{\partial x \partial z} A(x, z). \end{aligned} \tag{9}$$

In view of (9), boundary condition (4) is automatically satisfied. Boundary condition (5) will be satisfied if

$$A(x, z) = A(x, -z). \tag{10}$$

For $\alpha_1 \neq \pi/2$, boundary condition (3) will be satisfied if $A(x, z)$ has the form $e^{\pm ik \sec \alpha_1 x}$. This shows immediately that a unidirectionally conducting screen supports a wave traveling in the x-direction with a phase velocity $v_p = c \cos \alpha_1$ which is less than c, the velocity in free space. For a wave traveling in the direction of the negative x-axis, since

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2\right) A(x, z) = 0 \tag{11}$$

$$A(x, z) = A_0 e^{-ik \sec \alpha_1 x} \begin{cases} e^{-k \tan \alpha_1 z} & z > 0 \\ e^{k \tan \alpha_1 z} & z < 0 \end{cases}. \tag{12}$$

It is to be noted that the field components decay exponentially as $|z|$ increases. This surface wave is either "loosely bound" or "tightly bound" to the screen, depending on α_1 being small or large; there is no surface wave for $\alpha_1 = 0, \pi/2$. Using (9) and (12), the field components of the surface wave are written down explicitly as follows:

$$\begin{aligned} H_{\xi_1}(x, z) &= 0 \\ H_{\eta_1}(x, z) &= \mp e^{-ik \sec \alpha_1 x} e^{\mp k \tan \alpha_1 z} \\ H_z(x, z) &= i e^{-ik \sec \alpha_1 x} e^{\mp k \tan \alpha_1 z} \\ \boldsymbol{E}(x, z) &= \mp i \boldsymbol{H}(x, z). \end{aligned} \tag{13}$$

In (13), the upper sign if for $z>0$ and the lower sign $z<0$. An arbitrary value is used for the constant A_0 in (13). Notice further that

$$\begin{aligned} H_{\xi_1, \eta_1}(x, z) &= -H_{\xi_1, \eta_1}(x, -z) \\ E_{\xi_1, \eta_1}(x, z) &= E_{\xi_1, \eta_1}(x, -z) \\ H_z(x, z) &= H_z(x, -z) \\ E_z(x, z) &= -E_z(x, -z). \end{aligned} \tag{14}$$

The relations (14) result on account of the symmetry about the plane $z=0$. I is to be noticed that in view of (14), the equivalent to boundary condition (4) on the screen is

$$H_{\xi_1}(x, y, o) = 0. \tag{15}$$

It is now desired to examine the effect of terminating the surface waveguide $(0 \leq x \leq \infty)$ at $x=0$, on the surface wave given by (13) when it is incident from $x=\infty$. On account of the symmetry relations (14), it is enough to consider the region $z>0$.

Radiation from the Open End of the Surface Waveguide

No surface wave can be supported in the region $x<0$, and therefore the incident surface wave will be partly reflected back as a surface wave and partly converted into a radiation field. The current on the screen, and therefore the vector potential, are both only in the ξ_1 direction and no ξ_1 component of $\boldsymbol{H}$ is generated by the discontinuity. It is assumed that k has a small positive imaginary part ϵ, which is set equal to zero in the final formulas. All the field components may be obtained from the vector potential $A(x, z)$ which is related to the electric current density $i(x)$ by the formula

$$A(x,z) = A(x,-z) = \frac{i}{4}\int_0^\infty i(x')H_0^{(1)}[k\sqrt{(x-x')^2+z^2}]dx' \qquad z>0. \quad (16)$$

By making use of the following representations

$$E_{\xi_1}(x,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\zeta x}\bar{E}_{\xi_1}(\zeta,z)d\zeta \quad (17a)$$

$$i(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\zeta x}\bar{I}(\zeta)d\zeta \quad (17b)$$

$$A(x,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\zeta x}\bar{A}(\zeta,z)d\zeta, \quad (17c)$$

and (9), it follows for $z=0$ that

$$\bar{E}_{\xi_1}(\zeta,0) = \frac{i}{k}(k^2-\cos^2\alpha_1\zeta^2)\bar{A}(\zeta,0). \quad (18)$$

Since from (16)

$$\bar{A}(\zeta,z) = \frac{i}{4}\bar{I}(\zeta)\frac{2}{\sqrt{k^2-\zeta^2}}e^{i\xi z}, \quad (19)$$

where $\text{Im}\,\xi = \text{Im}\sqrt{k^2-\zeta^2}>0$, (18) reduces to

$$\bar{E}_{\xi_1}(\zeta,0) = -\frac{1}{2k}\cos^2\alpha_1[(k\sec\alpha_1)^2-\zeta^2]\frac{\bar{I}(\zeta)}{\sqrt{k^2-\zeta^2}}\cdot \quad (20)$$

It is first necessary to know the regions of regularity of the various transforms in (20). From (13) the incident total current density is obtained as

$$i^i(x) = 2e^{-ik\sec\alpha_1 x}. \quad (21)$$

Also as $x\to\infty$, $i(x)$ should obviously be of the form

$$i(x) = 2[e^{-ik\sec\alpha_1 x} + Re^{ik\sec\alpha_1 x}], \quad (22)$$

where the first term is the incident current density and R is the reflection coefficient at $x=0$. It is found from (22) that $[(k\sec\alpha_1)^2-\zeta^2]\bar{I}(\zeta)$ is regular in the lower half-plane ($\text{Im}\,\zeta<\epsilon$). Also $\bar{E}_{\xi_1}(\zeta,0)$ is regular in the upper half-plane ($\text{Im}\,\zeta>-\epsilon$). In addition, the transform of the Hankel function $2/\sqrt{k^2-\zeta^2}$ is regular and has no zero in the strip $|\text{Im}\,\zeta|<\epsilon$, and therefore the Wiener-Hopf procedure[6] can be applied to solve (20).

[6] B. Noble, "Methods Based on the Wiener-Hopf Techniques," Pergamon Press, Inc., New York, N. Y.; 1958.

Rewriting (20) as

$$[(k\sec\alpha_1)^2-\zeta^2]\bar{I}(\zeta)\frac{1}{\sqrt{k-\zeta}} = -2k\sec^2\alpha_1\sqrt{k+\zeta}\,\bar{E}_{\xi_1}(\zeta,0), \quad (23)$$

it is seen that the right-hand side of (23) is regular in the upper half-plane and the left-hand side is regular in the lower half-plane. Both are regular in the strip $|\text{Im}\,\zeta|<\epsilon$ and may be considered as analytic continuations of each other; together they define an integral function in the finite ζ-plane. By considering the asymptotic behavior of either side of (23) as $\zeta\to\infty$, the integral function defined by (23) may be shown to be a constant. By applying the Meixner corner condition, it is clear that the singularity of $E_{\xi_1}(x,0)$ at $x=0$ is the form $x^{-1/2}$, and hence $\bar{E}_{\xi_1}(\zeta,0)\sim\zeta^{-1/2}$ as $|\zeta|\to\infty$. From the right-hand side of (23), the integral function is seen to be a constant. Therefore, it is clear from (23) that $I(\zeta)\sim\zeta^{-3/2}$ as $\zeta\to\infty$, and hence, it follows that $i(x)$ vanishes at $x=0$ as $x^{1/2}$. This is in accordance with the Meixner corner condition and the requirement that the current at the end of the wires composing the screen should vanish.

From (23) and (17b), it follows that

$$i(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{D\sqrt{k-\zeta}}{[(k\sec\alpha_1)^2-\zeta^2]}e^{i\zeta x}d\zeta \quad (24)$$

where D is a constant to be determined, and the integration contour passes below both the poles $\zeta=\pm k\sec\alpha_1$. By closing the contour in the lower half-plane, it turns out that $i(x)=0$ for $x<0$, as it should. By closing it in the upper half-plane the value of $i(x)$ for $x>0$ can be evaluated. In particular, since the incident current density is contributed by the pole $\zeta=-k\sec\alpha_1$, it may be shown using (24) and (21) that

$$D = \frac{4ik^{1/2}\sec\alpha_1}{(1+\sec\alpha_1)^{1/2}}\cdot \quad (25)$$

Since the pole $\zeta=k\sec\alpha_1$ gives rise to the reflected current density, it is found with the help of (22), (23), and (25) that

$$R = -\left(\frac{1-\sec\alpha_1}{1+\sec\alpha_1}\right)^{1/2}\cdot \quad (26)$$

The magnitude of the reflection coefficient monotonically increases as α_1 increases.

With the help of (17c), (19) and (24), it follows that

$$A(x,z) = \frac{iD}{4\pi}\int e^{i\zeta x+i\xi z}\frac{1}{\sqrt{k+\zeta}(k^2\sec^2\alpha_1-\zeta^2)}d\zeta. \quad (27)$$

It is possible to express $A(x,z)$ in a closed form in terms of Fresnel integrals. However, since the interest is only in finding the radiation pattern, the expression for

$A(x, z)$, valid in the far zone, will be found. Introduce the polar coordinates

$$-x = \rho \cos\theta \quad z = \rho \sin\theta \quad (0 \leq \theta \leq \pi). \tag{28}$$

The path of integration in (27) is deformed by setting

$$\zeta = k \sin\tau \qquad \xi = k \cos\tau. \tag{29}$$

With (28) and (29), (27) reduces to

$$A(\rho, \theta) = \frac{iD}{4\pi k^{3/2}} \int \frac{\sqrt{1 - \sin\tau}}{(\sec^2\alpha_1 - \sin^2\tau)} e^{ik\rho \sin(\theta - \tau)} d\tau. \tag{30}$$

For $k\rho \gg 1$, (30) is evaluated by the method of stationary phase to yield

$$A(\rho, \theta) = \frac{\sec\alpha_1}{k(1 + \sec\alpha_1)^{1/2}} \left(\frac{2}{\pi k\rho}\right)^{1/2} \cdot e^{i(k\rho - (\pi/4))} \frac{(1 + \cos\theta)^{1/2}}{(\sec^2\alpha_1 - \cos^2\theta)} \cdot \tag{31}$$

Since $A(\rho, \theta)$ is in the ξ_1 direction, it follows from (1) and (28) that

$$\mathbf{A}(\rho, \theta) = [-\hat{\rho} \cos\theta \cos\alpha_1 + \hat{\theta} \sin\theta \sin\alpha_1 + \hat{y} \sin\alpha_1] A(\rho, \theta). \tag{32}$$

With the help of (32), the components of the field quantities are readily computed and the following result is obtained:

$$|H|^2 = \frac{2}{k\pi\rho} \frac{1}{(1 + \sec\alpha_1)} \frac{(1 + \cos\theta)}{(\sec^2\alpha_1 - \cos^2\theta)} \cdot \tag{33}$$

Hence, the total power radiated per unit width of the screen is

$$P_R = \int_{-\pi}^{\pi} |H|^2 \rho d\theta = \frac{4\cos^2\alpha_1}{k \tan\alpha_1 (1 + \cos\alpha_1)} \cdot \tag{34}$$

It is to be noted that (34) gives the power radiated in both the half-spaces $z \gtrless 0$. From (13), the total incident power flowing across unit width in the direction of propagation is obtained as

$$P_i = 2 \int_0^\infty |H^i|^2 \cos\alpha_1 dz = 4 \cos\alpha_1 \int_0^\infty e^{-2k \tan\alpha_1 z} dz = \frac{2 \cos\alpha_1}{k \tan\alpha_1} \cdot \tag{35}$$

The total power carried by the reflected surface wave per unit width of the screen is obtained using (26), as follows:

$$P_r = P_i |R|^2 = \frac{2 \cos\alpha_1}{k \tan\alpha_1} \left(\frac{1 - \cos\alpha_1}{1 + \cos\alpha_1}\right). \tag{36}$$

Using (34) and (36), it results that

$$P_R + P_n = \frac{2 \cos\alpha_1}{k \tan\alpha_1} = P_i. \tag{37}$$

Hence, it is seen that the total incident power per unit width of the screen is equal to the sum of the power carried by the reflected surface wave and the power converted into the radiation field.

The radiation pattern as given in (33) is plotted in Fig. 1 for four different values of α_1. It is seen to consist of a single lobe with its null in the direction of the surface waveguide and its maximum in the direction of the geometrical extension of the waveguide. Besides, the beamwidth is seen to reduce as α_1 is decreased, as is to be expected from the behavior of the transverse attenuation in (12). Also, it is obvious from (34) and (35) that the proportion of the incident power that is radiated decreases monotonically, at first slowly and then more rapidly, as α_1 is increased. If the general features of radiation of the limiting case are also true for the helical guide, then it follows that as the pitch is increased, both the reflection at the open end and the beamwidth of the pattern are reduced for the corresponding mode of excitation.

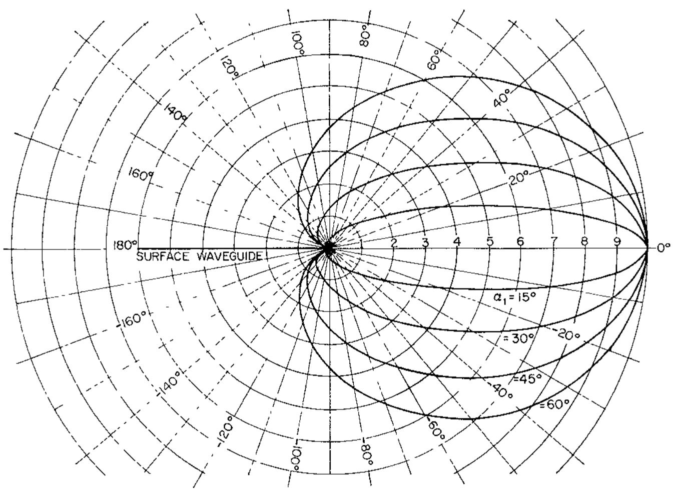


Fig. 1—Radiation diagram of the surface waveguide with open end.

Radiation from Discontinuity Formed by the Junction of Two Surface Waveguides

Another unidirectionally conducting semi-infinite screen is now considered to occupy the region $(-\infty \leq x \leq 0,\ -\infty \leq y \leq \infty,\ z = 0)$ and is joined along $x = 0$ to the first surface waveguide $(0 \leq x \leq \infty,\ -\infty \leq y \leq \infty,\ z = 0)$. The second semi-infinite screen $(-\infty \leq x \leq 0)$ is assumed to be conducting in the direction ξ_2 and insulating in the perpendicular direction η_2 where ξ_2, η_2 are given in (1). As before, the surface wave given by (13) is assumed to be incident from $x = \infty$. At the discontinuity $x = 0$, a part of the incident surface wave is reflected, another part transmitted as a surface wave and the remaining energy in the incident surface wave is converted into a radiation field.

A general solution of Maxwell's equations (2) can be obtained[7] as the sum of two independent solutions E_1 and E_2 such that

$$\text{Type I: } E_1 = -iH_1$$
$$\text{Type II: } E_1 = iH_1. \quad (38)$$

From (3)–(5), and (15), the boundary conditions for $z=0$ become

$$\hat{\xi}_1\cdot \boldsymbol{E}_1 = 0 \text{ for } x > 0; \quad \hat{\xi}_2\cdot \boldsymbol{E}_1 = 0 \text{ for } x < 0$$
$$\hat{\xi}_1\cdot \boldsymbol{E}_2 = 0 \text{ for } x > 0; \quad \hat{\xi}_2\cdot \boldsymbol{E}_2 = 0 \text{ for } x < 0. \quad (39)$$

For the incident field it is seen that

$$E^i = -iH^i \quad \text{for } z > 0$$
$$E^i = iH^i \quad \text{for } z < 0. \quad (40)$$

Since E_1 and E_2 are separated in the boundary conditions and since the fields preserve the symmetry of the wavetype [namely E_1 or E_2] of the incident wave, it follows that for the scattered fields also

$$E^s = -iH^s \quad \text{for } z > 0$$
$$E^s = iH^s \quad \text{for } z > 0. \quad (41)$$

Again the symmetry about $z=0$ [14] permits the detailed consideration of only the region $z>0$.

The incident field as well as the geometry of the problem is independent of the y-coordinate and, hence, all the components of the scattered field likewise are independent of the y-coordinate and are therefore derived conveniently using the y-component of the electric and magnetic fields. In view of (41), the entire scattered field may be derived from the y-component of the magnetic field only, using the following relations which are easily derived from (2):

$$E_x^s(x, z) = \frac{1}{ik}\frac{\partial}{\partial z} H_y^s(x, z)$$
$$E_z^s(x, z) = -\frac{1}{ik}\frac{\partial}{\partial x} H_y^s(x, z)$$
$$H_x^s(x, z) = \frac{1}{k}\frac{\partial}{\partial z} H_y^s(x, z)$$
$$H_z^s(x, z) = -\frac{1}{k}\frac{\partial}{\partial x} H_y^s(x, z) \quad (42)$$

The sum of the incident and the scattered fields, denoted by the superscripts i and s respectively, is the total field. Since from (2)

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2\right] H_y^s(x, z) = 0, \quad (43)$$

$H_y^s(x, z)$ may be assumed as follows:

$$H_y^s(x, z) = \frac{1}{2\pi}\int f(\zeta) e^{i\zeta x + i\xi z} d\zeta \quad (44)$$

where $\text{Im } \xi = \text{Im } \sqrt{k^2-\zeta^2} > 0$.

In view of (13) and (15), the boundary conditions [(3)–(5)] on the screens become

$$H_{\xi_1}^s(x, 0) = 0 \quad \text{for } x > 0 \quad (45a)$$
$$H_{\xi_2}^s(x, 0) = -H_{\xi_2}^i(x, 0) \quad \text{for } x < 0. \quad (45b)$$

Using (1), (42) and (44), it is found that

$$H_{\xi_{1,2}}^s(x, z) = \frac{1}{2\pi}\int \left[i \cos\alpha_{1,2} \frac{\xi}{k} + \sin\alpha_{1,2}\right] f(\zeta) e^{i\zeta x + i\xi z} d\zeta. \quad (46)$$

From (46) and (45a), it is seen that

$$\left[i \cos\alpha_1 \frac{\xi}{k} + \sin\alpha_1\right] f(\zeta) = u^+(\zeta) \quad (47)$$

where $u^+(\zeta)$ is regular in the upper half-plane $\text{Im } \zeta > -\epsilon$. For $x<0$, it follows from (46) and (13) that

$$H_{\xi_2}(x, z) = H_{\xi_2}^s(x, z) + H_{\xi_2}^i(x, z)$$
$$= \frac{1}{2\pi}\int \left[i \cos\alpha_2 \frac{\xi}{k} + \sin\alpha_2\right] f(\zeta) e^{i\zeta x + i\xi z} d\zeta$$
$$- \sin(\alpha_2 - \alpha_1) e^{-ik \sec\alpha_1 x - k \tan\alpha_1 z}. \quad (48)$$

Rewriting (48) for $z=0$ as

$$H_{\xi_2}^s(x, 0) + H_{\xi_2}^i(x, 0)$$
$$= \frac{1}{2\pi}\int \left[\left\{i \cos\alpha_2 \frac{\xi}{k} + \sin\alpha_2\right\} f(\zeta) - \frac{i \sin(\alpha_2 - \alpha_1)}{\zeta + k \sec\alpha_1}\right] e^{i\zeta x} d\zeta \quad (49)$$

and using (45b), it may be argued that

$$\left[i \cos\alpha_2 \frac{\xi}{k} + \sin\alpha_2\right] f(\zeta) - \frac{i \sin(\alpha_2 - \alpha_1)}{\zeta + k \sec\alpha_1} = L^-(\zeta) \quad (50)$$

where $L^-(\zeta)$ is regular in the lower half-plane $\text{Im } \zeta < \epsilon$. Eliminating $f(\zeta)$ from (47) and (50), it is found that

$$u^+(\zeta) \frac{\cos\alpha_2}{\cos\alpha_1} \frac{K_2(\zeta)}{K_1(\zeta)} - \frac{i \sin(\alpha_2 - \alpha_1)}{\zeta + k \sec\alpha_1} = L^-(\zeta) \quad (51)$$

where

$$K_{1,2}(\zeta) = \left[1 - \frac{k \tan\alpha_{1,2}}{\sqrt{\zeta^2 - k^2}}\right]. \quad (52)$$

The transform relation (51) is valid in the strip $|\text{Im } \zeta| < \epsilon$. The zeros of $K_{1,2}(\zeta)$ lie outside this strip. The

[7] V. H. Rumsey, "A new way of solving Maxwell's equations," IRE Trans. on Antennas and Propagation, vol. AP-9, pp. 461–465; September, 1961.

standard Wiener-Hopf procedure requires the splitting up of the functions in (52) in the form

$$K_1(\zeta) = \frac{K_1^+(\zeta)}{K_1^-(\zeta)}; \qquad K_2(\zeta) = \frac{K_2^+(\zeta)}{K_2^-(\zeta)} \tag{53}$$

where

$$K_{1,2}^{\pm}(\zeta) = \exp\left\{-\frac{1}{2\pi i}\int_{-\infty\mp(i\epsilon/2)}^{\infty\mp(i\epsilon/2)} \frac{\log\left[1-\dfrac{k\tan\alpha_{1,2}}{\sqrt{t^2-k^2}}\right]}{\zeta - t}\,dt.\right. \tag{54}$$

The $+$ and $-$ functions are regular and not zero in the upper and lower half-planes respectively. Using (53) and rewriting (51) as

$$u^+(\zeta)\frac{\cos\alpha_2}{\cos\alpha_1}\frac{K_2^+(\zeta)}{K_1^+(\zeta)} - \frac{i\sin(\alpha_2-\alpha_1)}{\zeta + k\sec\alpha_1}\frac{K_2^-(-k\sec\alpha_1)}{K_1^-(-k\sec\alpha_1)}$$
$$= L^-(\zeta)\frac{K_2^-(\zeta)}{K_1^-(\zeta)} - \frac{i\sin(\alpha_2-\alpha_1)}{\zeta + k\sec\alpha_1} \cdot\left[\frac{K_2^-(-k\sec\alpha_1)}{K_1^-(-k\sec\alpha_1)} - \frac{K_2^-(\zeta)}{K_1^-(\zeta)}\right], \tag{55}$$

it is seen that the left and the right sides are respectively regular in the upper and the lower half-planes. Both sides are regular in the strip $|\mathrm{Im}\,\zeta| < \epsilon$ and may be considered as analytic continuations of each other; together they define an integral function in the finite ζ-plane. For $|\zeta|\to\infty$, $u^+(\zeta)$ is $0(\zeta^{-\delta})$ where $\delta > 0$ in order that the integrals in (46) converge when $z=0$. Also, it can be shown that the factor $K_{1,2}^+(\zeta)$ are $0(1)$ as $|\zeta|\to\infty$. Hence, by Liouville's theorem, the integral function defined by (55) is zero. Equating the left side of (55) to zero, an expression for $u^+(\zeta)$ is obtained, and, using it in (47) and (44), it readily follows that

$$H_y^s(x, z) = \frac{ik\sin(\alpha_2-\alpha_1)}{2\pi\cos\alpha_2}\int_{-\infty}^{\infty}\frac{d\zeta e^{i\zeta x + i\xi z}}{[k\tan\alpha_1 - \sqrt{\zeta^2-k^2}][\zeta + k\sec\alpha_1]} \cdot\frac{K_2^-(-k\sec\alpha_1)}{K_1^-(-k\sec\alpha_1)}\frac{K_1^+(\zeta)}{K_2^+(\zeta)}. \tag{56}$$

Expressions for the transmitted and reflected surface waves and the radiation field may be obtained by evaluating (56) asymptotically for large x. From (54), it follows that

$$K_{1,2}^+(\zeta) = \frac{1}{K_{1,2}^-(-\zeta)} \tag{57}$$

and that $K_{1,2}^-(\zeta)$ is analytic except for the branch points $\pm k$ and the logarithmic singularities at $\pm k\sec\alpha_{1,2}$. The integrand in (54) is $0(1/|t|^2)$ for large $|t|$, and, hence, the integration contour $[-\infty + i\epsilon/2$ to $\infty + i\epsilon/2]$ may therefore be deformed into a new one embracing the radial branch cut from k to ∞. This contour may be deformed slightly at any point except possibly at $\zeta = k$ and $\zeta = k\sec\alpha_{1,2}$ where the singularities of the integrand occur. Hence, $K_{1,2}^-(\zeta)$ is analytic and nonzero everywhere except possibly at k and $k\sec\alpha_{1,2}$. In view of (57), $K_{1,2}^+(\zeta)$ is regular and nonzero everywhere except at $\zeta = -k$, $-k\sec\alpha_{1,2}$. It is evident from (52) and (53) that

$$K_{1,2}^-(\zeta) = \frac{K_{1,2}^+(\zeta)\sqrt{\zeta^2-k^2}}{\sqrt{\zeta^2-k^2} - k\tan\alpha_{1,2}}. \tag{58}$$

Since $K_{1,2}^+(\zeta)$ is regular and zero at $\zeta = k$, it follows from (58) that $K_{1,2}^-(\zeta)$ has a branch point at $\zeta = k$. Again because $K_{1,2}^+(\zeta)$ is regular and nonzero at $\zeta = k\sec\alpha_{1,2}$, $K_{1,2}^-(\zeta)$ has a simple pole at $\zeta = k\sec\alpha_{1,2}$. In a similar fashion, it follows from (58) that $K_{1,2}^+(\zeta)$ has a branch point at $\zeta = -k$ and a zero at $\zeta = -k\sec\alpha_{1,2}$. The integrand in (56) has, therefore, simple poles at $\zeta = \pm k\sec\alpha_1$, $-k\sec\alpha_2$ and branch points at $\zeta = \pm k$. For x negative, (56) is evaluated by deforming the contour to a line parallel to, and just below, the original contour along the real axis [note that ϵ has been set equal to zero], with indentations above the singularities of the integrand which occur at $\zeta = -k$, $-k\sec\alpha_1$ and $-k\sec\alpha_2$. The poles at $\zeta = -k\sec\alpha_{1,2}$ give rise to the surface wave, whereas the singularity at $\zeta = -k$ gives the radiation field which decays as $1/|x|^{1/2}$ for large x. Hence, for large x, only the surface-wave contributions dominate and the evaluation of the contribution of (56) at the poles $\zeta = -k\sec\alpha_{1,2}$ yields

$$H_y^s(x, z) = \cos\alpha_1 e^{-ik\sec\alpha_1 x - k\tan\alpha_1 z} + \cos\alpha_1\frac{K_2^-(-k\sec\alpha_1)}{K_1^-(-k\sec\alpha_1)}\frac{K_1^+(-k\sec\alpha_2)}{K_2^-(-k\sec\alpha_2)} \cdot\frac{\tan^2\alpha_2}{\sec\alpha_2(\sec\alpha_1 - \sec\alpha_2)}\cdot e^{-ik\sec\alpha_2 x - k\tan\alpha_2 z}. \tag{59}$$

From (13) and (1), it is obvious that

$$H_y^i(x, z) = -\cos\alpha_1 e^{-ik\sec\alpha_1 x - k\tan\alpha_1 z}. \tag{60}$$

Using (57), (59) and (60), the total transmitted surface wave for $x<0$ is obtained as

$$[H_y(x, z)]_t = H_y^s(x, z) + H_y^i(x, z) = \frac{\cos\alpha_1 K_2^-(-k\sec\alpha_1)}{K_1^-(k\sec\alpha_2)K_1^-(-k\sec\alpha_1)}\frac{1}{K_2^-(-k\sec\alpha_2)} \cdot\frac{\tan^2\alpha_2}{\sec\alpha_2(\sec\alpha_1 - \sec\alpha_2)}e^{-ik\sec\alpha_2 x - k\tan\alpha_2 z}. \tag{61}$$

Notice that the incident wave (60) completely nullifies the surface wave with the value of $k\tan\alpha_1$ for the attenuation factor in the z-direction. This should be the case, since for $x<0$ the screen is conducting in the ξ_2 direction and hence can support only a surface wave with an attenuation factor $k\tan\alpha_2$. It may be easily shown that as $\alpha_2 \to \alpha_1$

$$[H_y(x, z)]_t = -\cos\alpha_1 e^{-ik\sec\alpha_1 x - k\tan\alpha_1 z} = H_y^i(x, z). \quad (62)$$

This too should be the case, since the incident wave will then be transmitted as it is without any disturbance.

For $x > 0$, (56) is evaluated by deforming the original contour along the real axis to one parallel to it slightly in the upper half-plane and indented below at the singularities $\zeta = k$ and $k \sec\alpha_1$. As before, the contribution of the integral (56) in the neighborhood of the singularity $\zeta = k$ gives rise to the radiation field, which for large x is small compared to the surface wave term. Evaluating the contribution of the integral near the pole gives the reflected surface wave

$$[H_y(x, y)]_r = \frac{\sin(\alpha_2 - \alpha_1)\tan\alpha_1}{2\cos\alpha_2\sec^2\alpha_1}\left[\frac{K_2^-(-k\sec\alpha_1)}{K_1^-(-k\sec\alpha_1)}\right]^2 \cdot e^{ik\sec\alpha_1 x - k\tan\alpha_1 z}. \quad (63)$$

The radiation field is obtained by substituting (28) and (29) in (56) and evaluating the resulting integral by the method of stationary phase for $k\rho \gg 1$. The result when (52) is made use of is

$$[H_y(x, z)]_R = \frac{1}{\sqrt{2\pi k\rho}} e^{i(k\rho + (\pi/4))} \frac{\sin(\alpha_2 - \alpha_1)}{\cos\alpha_2} \frac{K_2^-(-k\sec\alpha_1)}{K_1^-(-k\sec\alpha_1)} \times \frac{\sin\theta}{[\tan\alpha_1 + i\sin\theta]} \frac{1}{[\sec\alpha_1 - \cos\theta]} \frac{K_2^-(k\cos\theta)}{K_1^-(k\cos\theta)} \cdot \quad (64)$$

The subscript R denotes the radiation field. Note that when $\alpha_1 = \alpha_2$, the radiation field goes to zero as it should.

It remains only to determine $K_{1,2}^-(\zeta)$ from (54) which has been evaluated by Kay[4] in a different connection. In what follows only the expressions for $|K_{1,2}^-(-\zeta)|^2$ will be needed and this is taken from Kay's paper:

$$|K_{1,2}^-|^2 = \begin{cases} \left|\dfrac{\zeta - k}{\zeta - k\sec\alpha_{1,2}}\right| \left|\dfrac{\sqrt{\zeta^2 - k^2} + k\tan\alpha_{1,2}}{\sqrt{\zeta^2 - k^2} - k\tan\alpha_{1,2}}\right| & \zeta > k \\ \left|\dfrac{\zeta - k}{\zeta - k\sec\alpha_{1,2}}\right| & \zeta < k \end{cases}. \quad (65)$$

It is desired to find expressions for the power reflection coefficient R, the power transmission coefficient T and a coefficient S for the radiated power which denote respectively the proportion of the incident power that is reflected, transmitted and radiated. The total power in the reflected surface wave per unit width of the guide is obtained from (63), (42) and (65) as

$$P_r = 2\,\text{Re}\int_0^\infty \hat{x}\cdot \boldsymbol{E}_r \times \boldsymbol{H}_r^* dz = \frac{2\sin^2(\alpha_2 - \alpha_1)\sin\alpha_1}{k\cos^2\alpha_2(\sec\alpha_2 + \sec\alpha_1)^2} \cdot \quad (66)$$

The total power in the transmitted surface wave is calculated from (61), (42) and (65) as

$$P_t = 2\,\text{Re}\int_0^\infty \hat{x}\cdot \boldsymbol{E}_t \times \boldsymbol{H}_t^* dz = \frac{8\cos\alpha_1\tan\alpha_2}{k(\tan\alpha_2 + \tan\alpha_1)^2} \cdot \quad (67)$$

The power radiated per unit width of the screen per unit area in the direction θ is obtained from (64), (42) and (28) for $k\rho \gg 1$ as

$$S = \text{Re}\,\hat{\rho}\cdot \boldsymbol{E}_R \times \boldsymbol{H}_R^* = \frac{2}{\pi k}\frac{\sin^2(\alpha_2 - \alpha_1)}{\cos^2\alpha_2}\frac{\sec\alpha_1}{(\sec\alpha_2 + \sec\alpha_1)} f(\theta) \quad (68)$$

where the radiation pattern $f(\theta)$ is given by

$$f(\theta) = \sin^2\theta\{(\tan^2\alpha_1 + \sin^2\theta)(\sec\alpha_1 - \cos\theta) \cdot (\sec\alpha_2 - \cos\theta)\}^{-1}. \quad (69)$$

Hence, the total radiated power is

$$P_R = \int_0^{2\pi} S\rho d\theta = \frac{4\cos^2\alpha_1\sin\alpha_1}{k\tan^2\alpha_1(\tan\alpha_2 + \tan\alpha_1)^2} \cdot \left[\tan^2\alpha_1\sec\alpha_1\sec\alpha_2 - \tan\alpha_1\tan\alpha_2\sec^2\alpha_1 + \frac{\tan^2\alpha_2}{2} - \frac{\tan^2\alpha_1}{2}\right]. \quad (70)$$

From (35), (66), (67) and (70) R, T and S are obtained as

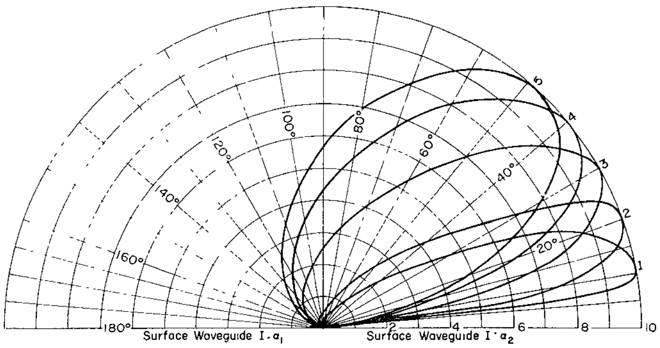
$$R = \sin^2\alpha_1 \frac{(\sec\alpha_2 - \sec\alpha_1)^2}{(\tan\alpha_2 + \tan\alpha_1)^2} \quad (71)$$

$$T = 1 - \frac{(\tan\alpha_2 - \tan\alpha_1)^2}{(\tan\alpha_2 + \tan\alpha_1)^2} \quad (72)$$

and

$$S = \frac{2\cos^2\alpha_1}{[\tan\alpha_2 + \tan\alpha_1]^2}\left[\tan^2\alpha_1\sec\alpha_1\sec\alpha_2 - \tan\alpha_1\tan\alpha_2\sec^2\alpha_1 + \frac{\tan^2\alpha_2}{2} - \frac{\tan^2\alpha_1}{2}\right]. \quad (73)$$

It is easily verified that $R + T + S = 1$ as it should. The radiation pattern (69) is plotted in Fig. 2 for several values of α_1 and α_2. It is noticed that it has a null in the plane of the waveguide, and that the beamwidth of the radiation pattern increases as the maximum of the pattern moves away from the plane of surface waveguide.



1) $\alpha_1 = 15°$, $\alpha_2 = 30°$
2) $\alpha_1 = 30°$, $\alpha_2 = 15°$
3) $\alpha_1 = 30°$, $\alpha_2 = 60°$
4) $\alpha_1 = 60°$, $\alpha_2 = 30°$
5) $\alpha_1 = 60°$, $\alpha_2 = 45°$

Fig. 2—Radiation diagram of the surface waveguide for various values of a_1 and a_2.

Radiation from Discontinuity Formed by the Junction of a Surface Waveguide and a Perfectly Conducting Half-Plane

The surface waveguide in the region $(-\infty \leq x \leq 0, -\infty \leq y \leq \infty, z=0)$ is now assumed to be replaced by a perfectly conducting half-plane and the incident surface wave is the same as is given by (13). As before, the incident, and hence the scattered fields, are independent of the y-coordinate; therefore, it follows from (2) that

$$E_x^s = \frac{1}{ik}\frac{\partial}{\partial z}H_y^s \qquad H_x^s = -\frac{1}{ik}\frac{\partial}{\partial z}E_y^s$$

$$E_z^s = -\frac{1}{ik}\frac{\partial}{\partial x}H_y^s \qquad H_z^s = \frac{1}{ik}\frac{\partial}{\partial x}E_y^s. \tag{74}$$

One of the boundary conditions on the screen is that

$$E_{\xi_1}(x, 0) = 0 \qquad -\infty < x < \infty. \tag{75}$$

Since from (13), $E_{\xi_1}{}^i(x, 0)=0$, it is obvious that $E_{\xi_1}{}^s(x, z)=0$. Therefore, from (1) and (74), it follows that

$$E_y^s(x, z) = -\frac{\cot\alpha_1}{ik}\frac{\partial}{\partial z}H_y^s(x, z). \tag{76}$$

With the representation (44) for $H_y^s(x, z)$, the following relations may be derived using (74), (76) and (1):

$$E_{\eta_1}{}^s(x, z) = \frac{1}{2\pi}\int -\frac{\xi_1}{k\sin\alpha_1}f(\zeta)e^{i\zeta x+i\xi z}d\zeta \tag{77}$$

$$H_{\xi_1}{}^s(x, z) = \frac{1}{2\pi}\int\frac{\cos^2\alpha_1}{k^2\sin\alpha_1}[k^2\sec^2\alpha_1 - \zeta^2]f(\zeta) \cdot e^{i\zeta x+i\xi z}d\zeta. \tag{78}$$

The remaining boundary conditions on the screens are

$$H_{\xi_1}{}^s(x, 0) = 0 \qquad \text{for } x > 0 \tag{79}$$

$$E_{\eta_1}{}^s(x, 0) = -ie^{-ik\sec\alpha_1 x} \qquad \text{for } x < 0. \tag{80}$$

In view of (78) and (79), it follows that

$$[k^2\sec^2\alpha_1 - \zeta^2]f(\zeta) = u^+(\zeta), \tag{81}$$

where $u^+(\zeta)$ is regular in the upper half-plane $\operatorname{Im}\zeta > -\epsilon$. In a similar manner from (77) and (80), it may be shown that

$$\frac{\xi}{k\sin\alpha_1}f(\zeta) + \frac{1}{\zeta + k\sec\alpha_1} = L^-(\zeta), \tag{82}$$

where $L^-(\zeta)$ is regular in the lower half-plane $\operatorname{Im}\zeta < \epsilon$. The transform relations are regular in the strip $|\operatorname{Im}\zeta| < \epsilon$, and hence the Wiener-Hopf procedure may be applied. By substituting for $f(\zeta)$ in (82) from (81) and rearranging the resulting expression, it follows that

$$\frac{\sqrt{k+\zeta}\,u^+(\zeta)}{k\sin\alpha_1(k\sec\alpha_1+\zeta)} + \frac{2k\sec\alpha_1}{(\zeta+k\sec\alpha_1)\sqrt{k+k\sec\alpha_1}} = \frac{(k\sec\alpha_1-\zeta)L^-(\zeta)}{\sqrt{k-\zeta}} + \frac{1}{\zeta+k\sec\alpha_1} \cdot\left[\frac{k\sec\alpha_1}{\sqrt{k-\zeta}} - \frac{2k\sec\alpha_1}{\sqrt{k+k\sec\alpha_1}}\right]. \tag{83}$$

By the arguments of the Wiener-Hopf procedure (83) may be shown to define an integral function which is actually zero. Consequently from (83), (81) and (76), it is found that

$$H_y^s(x, z) = \frac{1}{2\pi}\int\frac{2k^2\tan\alpha_1 d\zeta e^{i\zeta x+i\xi z}}{\sqrt{k+k\sec\alpha_1}\sqrt{k+\zeta}(\zeta^2-k^2\sec^2\alpha_1)}\cdot \tag{84}$$

For x negative, (84) is evaluated by deforming the contour to a line parallel to the original contour along the real axis, slightly in the lower half-plane and indented above the singularities of the integrand which occur at $\zeta=-k$, $-k\sec\alpha_1$. The singularity at $\zeta=-k$ gives the radiation field which decays as $1/|x|^{1/2}$ for large x and, hence, for large negative x, the significant contribution arises due to the pole $\zeta=-k\sec\alpha_1$. Evaluation of the residue of the integral at the pole $\zeta=-k\sec\alpha_1$ gives

$$[H_y^s(x, z)]_t = \cos\alpha_1\,e^{-ik\sec\alpha_1 x - k\tan\alpha_1 z}. \tag{85}$$

Notice that (85) is exactly cancelled by the incident field (60). This should be the case, since a perfectly conducting half-plane cannot support a surface wave of the type (85).

For $x>0$, (84) is evaluated by deforming the original contour along the real axis, to one parallel to it slightly in the upper half-plane and indented below the singularity at $\zeta=k\sec\alpha_1$. This pole gives rise to the reflected surface wave and its value is

$$[H_y(x, z)]_r = \frac{i\sin\alpha_1}{(1+\sec\alpha_1)}e^{ik\sec\alpha_1 x - k\tan\alpha_1 z}. \tag{86}$$

To obtain the radiation field, (28) and (29) are substituted in (84), and the resulting integral is evaluated by the method of stationary phase for $k\rho \gg 1$. The result is

$$[H_y(x, z)]_R = \left(\frac{2}{\pi k \rho}\right)^{1/2} e^{i(k\rho - (\pi/4))} \cdot \frac{\tan \alpha_1}{(1 + \sec \alpha_1)^{1/2}} \frac{(1 + \cos \theta)^{1/2}}{(\cos^2 \theta - \sec^2 \alpha_1)} \cdot \quad (87)$$

The total power in the reflected surface wave per unit width of the screen is easily computed from (86), (74) and (76) as

$$P_r = 2 \int_0^\infty \hat{x} \cdot \boldsymbol{E}_r \times \boldsymbol{H}_r^* dz = \frac{2 \sin \alpha_1}{k(1 + \sec \alpha_1)^2} \cdot \quad (88)$$

The power radiated per unit width of the screen, per unit area in the direction θ is obtained from (87), (74), (76) and (28) when $k\rho \gg 1$ as

$$S = \text{Re}\, \hat{\rho} \cdot \boldsymbol{E}_R \times \boldsymbol{H}_R^* = \frac{2}{\pi k \rho} \frac{1}{(1 + \sec \alpha_1)} \frac{1 + \cos \theta}{(\sec^2 \alpha_1 - \cos^2 \theta)} \cdot \quad (89)$$

Hence, the total radiated power is

$$P_R = \int_0^{2\pi} S \rho d\theta = \frac{4}{k} \frac{1}{(1 + \sec \alpha_1)} \frac{\cos \alpha_1}{\tan \alpha_1} \cdot \quad (90)$$

It is to be noted that $P_r + P_R$ is equal to P_i as given in (35). The power reflection coefficient and the radiation pattern are noticed to be the same with or without the terminating perfectly conducting half-plane.

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Surface Waves on Radially Inhomogeneous Cylinders*

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Summary—A characteristic equation and a cutoff equation are derived for higher order surface-wave modes on lossless isotropic cylinders with arbitrary radial permittivity variation. The derivation, based on the use of the fundamental matrix of a set of differential equations, reduces analytical work and results in expressions well suited for digital computer evaluation of surface-wave eigenvalues and mode spectra. The theory is applied in an investigation of HE_{21} and EH_{21} mode propagation for a particular set of models for the radially varying permittivity. Typical results showing eigenvalue variation, dispersion characteristics and radial field variation, including experimental verification of dispersion characteristics, are shown. The method of analysis can be extended to anisotropic cylinders with permittivity a function of both radius and frequency.

INTRODUCTION

THIS PAPER is concerned with the problem of surface-wave propagation along lossless isotropic cylinders with radial permittivity variation. The permittivity variation may be described by a function of the radius or an experimental curve, with discontinuities allowed. Step permittivity variation, such as that created by dielectric rods and tubes made of constant permittivity material,[1,2] constitutes a special case of the problem.

Electromagnetic-wave propagation along cylindrical structures inhomogeneous in the transverse plane has been investigated by Adler.[3,4] Some basic results about orthogonality, power flow and phase constants are obtained but the general problem of formulating the differential equations and obtaining their solutions is not considered. An interesting way to formulate differential equations for fields in inhomogeneous media has been proposed by Smith.[5] The approach is based on a transformation which contains the space dependent permittivity. This involves work with quantities other than electromagnetic fields and may not be desirable in a

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